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AERSP 304

5/4/22

Honors Option Report

Overview:

This honors option project was aimed at analyzing the three-dimensional three-body problem. The two-dimensional three-body problem was first introduced in project 1 of AERSP 304. The 3D version of the project added a z equation of motion. This equation created motion in all three dimensions. However, later analysis shows that the x and y motion are coupled, while z motion is dependent purely on itself. This project also got a look at halo orbits, found about the L2 Lagrange Point.

Methodology:

First, the graph of the three-body problem was analyzed. The r position vector could then be found, for both r1 and r2, which then combines into a singular r. The derivative of this vector was then taken, creating a velocity vector. This could then be used to find the kinetic energy of the system.

Using the given gravitational potential energy, the three Lagrange equations could then be found, for x, y, and z. The three equations of motion were then time scaled using omega. These are the non-linear equations of motion.

These time-scaled equation could then be linearized, using linearization.

The Lagrange points of the system were then found, using the non-linear equations. The assumed conditions of a Lagrange points are that all velocities and accelerations are zero. From these conditions, it was found that z is always zero. Thus, the Lagrange points are planar. That said, the Lagrange point coordinates for the 3D three-body problem are the same as the 2D coordinates, except the additional z equals 0 point.

Using the z equals zero condition, the linearized equations can now be adjusted. This decouples the z equation of motion from the x and y equations of motion.

A state-space was then made of the positions and velocities to reduce the equations to first order. This creates a six-by-six matrix that visualizes the coupling of the x and u equations and the individuality of the z equations (i.e. decoupled from x and y).

Finally, the z initial condition was found. Using the solution to the

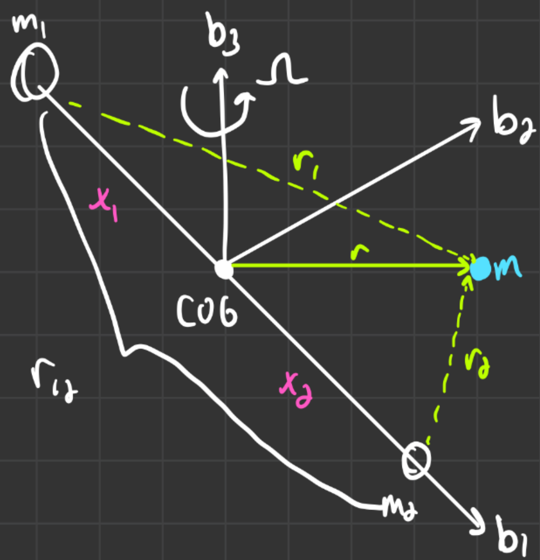
$$z'' + kx = 0$$

equation, the frequency could be found. A ‘guess’ condition was used, with an amplitude slightly above the radius of the moon and a phase shift of 30 degrees. This led to both z and z velocity initial conditions for both L2 and L4.

The x and y initial conditions were not found mathematically. However, most of the original initial conditions used for Project 1 were used.

The outlier is the x position initial condition for the L2 Lagrange point. With a halo orbit motion visualized, a guess and check method was utilized to get the circulation about x and y to vary minimally in y. This led to an x initial condition of 1.0895.

The entirety of the derivation process can be seen below.



$$\Omega = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{\sqrt{\mu}} r_{12}^{3/2}$$

$$\vec{\Omega} = \Omega \hat{b}_3$$

$$\Omega = 2\pi \left(\frac{2\pi}{\sqrt{\mu}} r_{12}^{3/2} \right)^{-1}$$

$$\Omega = \sqrt{\mu}\, r_{12}^{-3/2} = \sqrt{\frac{\mu}{r_{12}^3}}$$

$$\mu = G(m_1 + m_2), \quad m_1 x_1 + m_2 x_2 = 0$$

$$x_2 = x_1 + r_{12}$$

$$x_2 - x_1 = r_{12}$$

$$\pi_1 = \frac{m_1}{m_1 + m_2} \quad \pi_2 = \frac{m_2}{m_1 + m_2}$$

$$x_1 = -\pi_2 r_{12}$$

$$x_2 = \pi_1 r_{12}$$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{r}_1 = (x - x_1)\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{r}_1 = (x + \pi_2 r_{12})\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{r}_2 = (x - \pi_1 r_{12})\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{r} = x\hat{b}_1 + y\hat{b}_2 + z\hat{b}_3$$

$$\dot{\vec{r}} = \dot{x}\hat{b}_1 + x\dot{\hat{b}}_1 + \dot{y}\hat{b}_2 + y\dot{\hat{b}}_2 + \dot{z}\hat{b}_3 + \cancel{z\dot{\hat{b}}_3}^{\,0}$$

$$\dot{\hat{b}}_1 = \vec{\Omega} \times \hat{b}_1 = \langle 0, 0, \Omega\rangle \times \langle 1, 0, 0\rangle = \langle 0, \Omega, 0\rangle$$

$$\dot{\hat{b}}_2 = \vec{\Omega} \times \hat{b}_2 = \langle 0, 0, \Omega\rangle \times \langle 0, 1, 0\rangle = \langle -\Omega, 0, 0\rangle$$

$$\dot{\vec{r}} = \dot{x}\hat{b}_1 + x\Omega\hat{b}_2 + \dot{y}\hat{b}_2 - y\Omega\hat{b}_1 + \dot{z}\hat{b}_3$$

$$= (\dot{x} - y\Omega)\hat{b}_1 + (\dot{y} + x\Omega)\hat{b}_2 + \dot{z}\hat{b}_3$$

$$T = \tfrac{1}{2} m_3 \left(\dot{\vec{r}} \cdot \dot{\vec{r}} \right)$$

$$\rightarrow \dot{\vec{r}} \cdot \dot{\vec{r}} = (\dot{x} - y\Omega)^2 + (\dot{y} + x\Omega)^2 + \dot{z}^2$$

$$= \dot{x}^2 - 2\dot{x}y\Omega + y^2\Omega^2 + \dot{y}^2 + 2\dot{y}x\Omega + x^2\Omega^2 + \dot{z}^2$$

$$V = -\frac{G m_3 m_1}{r_1} - \frac{G m_3 m_2}{r_2}$$

$$L = T - V$$

$$= \frac{m_3}{2}\left(\dot{x}^2 - 2\dot{x}y\Omega + y^2\Omega^2 + \dot{y}^2 + 2\dot{y}x\Omega + x^2\Omega^2 + \dot{z}^2\right) + \frac{G m_3 m_1}{r_1} + \frac{G m_3 m_2}{r_2}$$

$$= \frac{m_3}{2}\left(\dot{x}^2 + \dot{y}^2 + 2\Omega(x\dot{y} - y\dot{x}) + \Omega^2(y^2 + x^2) + \dot{z}^2\right) + G m_3 \left(\frac{m_1}{\sqrt{(x + \pi_2 r_{12})^2 + y^2 + z^2}} + \frac{m_2}{\sqrt{(x - \pi_1 r_{12})^2 + y^2 + z^2}} \right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial}{}\left(\frac{\partial L}{}\right) - \frac{\partial L}{} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{m_3}{2}\left(2\dot{x} - 2y\Omega\right) = m_3\left(\dot{x} - y\Omega\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_3\left(\ddot{x} - \dot{y}\Omega\right)$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{m_3}{2}\left(2\dot{y} + 2x\Omega\right) = m_3\left(\dot{y} + x\Omega\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) = m_3\left(\ddot{y} + \dot{x}\Omega\right)$$

$$\frac{\partial L}{\partial \dot{z}} = m_3\dot{z}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{z}}\right) = m_3\ddot{z}$$

$$\frac{\partial L}{\partial x} = \frac{m_3}{2}\left(2\Omega\dot{y} + 2\Omega^2 x\right) + Gm_3\left(-\tfrac{1}{2}m_1\left((x+\pi_2 r_{12})^2 + y^2 + z^2\right)^{-3/2} 2\left(x+\pi_2 r_{12}\right) - \tfrac{1}{2}m_2\left((x-\pi_1 r_{12})^2 + y^2 + z^2\right)^{-3/2} 2\left(x-\pi_1 r_{12}\right)\right)$$

$$= m_3\left[\Omega\dot{y} + \Omega^2 x - G\left(\frac{m_1}{r^3}\left(x+\pi_2 r_{12}\right) + \frac{m_2}{r^3}\left(x-\pi_1 r_{12}\right)\right)\right]$$

$$\frac{\partial L}{\partial \dot{z}} = m_3 \dot{z}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m_3 \ddot{z}$$

$$\frac{\partial L}{\partial x} = \frac{m_3}{2}\left(2\Omega\dot{y} + 2\Omega^2 x\right) + G m_3\left(-\tfrac{1}{2}m_1\left((x+\pi_2 r_{12})^2+y^2+z^2\right)^{-3/2} 2(x+\pi_2 r_{12}) - \tfrac{1}{2}m_2\left((x-\pi_1 r_{12})^2+y^2+z^2\right)^{-3/2} 2(x-\pi_1 r_{12})\right)$$

$$= m_3\left[\Omega\dot{y} + \Omega^2 x - G\left(\frac{m_1}{r_1^3}(x+\pi_2 r_{12}) + \frac{m_2}{r_2^3}(x-\pi_1 r_{12})\right)\right]$$

$$\frac{\partial L}{\partial y} - m_3\left[\Omega^2 y - \dot{x}\Omega - G y\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right)\right]$$

$$\frac{\partial L}{\partial z} = -G m_3 z\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right)$$

$$\mathcal{L}_1 = \cancel{m_3}\left(\ddot{x} - \dot{y}\Omega\right) - \cancel{m_3}\left[\Omega\dot{y} + \Omega^2 x - G\left(\frac{m_1}{r_1^3}(x+\pi_2 r_{12}) + \frac{m_2}{r_2^3}(x-\pi_1 r_{12})\right)\right] = 0$$

$$\mathcal{L}_2 - \cancel{m_3}\left(\ddot{y} + \dot{x}\Omega\right) - \cancel{m_3}\left[\Omega^2 y - \dot{x}\Omega - G y\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right)\right] = 0$$

$$\mathcal{L}_3 = \cancel{m_3}\ddot{z} + G\cancel{m_3} z\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right)$$

$$\mathcal{L}_1 = \cancel{m_3}\left(\ddot{x} - \dot{y}\Omega\right) - \cancel{m_3}\left[\Omega\dot{y} + \Omega^2 x - G\left(\frac{m_1}{r_1^3}(x+\pi_2 r_{12}) + \frac{m_2}{r_2^3}(x-\pi_1 r_{12})\right)\right] = 0$$

$$= \ddot{x} - \dot{y}\Omega - \Omega\dot{y} - \Omega^2 x = -G\left(\frac{m_1(x+\pi_2 r_{12})}{r_1^3} + \frac{m_2(x-\pi_1 r_{12})}{r_2^3}\right)$$

$$\boxed{= \ddot{x} - 2\dot{y}\Omega - \Omega^2 x = -\frac{\mu_1(x+\pi_2 r_{12})}{r_1^3} - \frac{\mu_2(x-\pi_1 r_{12})}{r_2^3}}$$

$$\mathcal{L}_2 = \cancel{m_3}\left(\ddot{y} + \dot{x}\Omega\right) - \cancel{m_3}\left[\Omega^2 y - \dot{x}\Omega - Gy\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right)\right] = 0$$

$$= \ddot{y} + \dot{x}\Omega + \dot{x}\Omega - \Omega^2 y = -Gy\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right)$$

$$\boxed{= \ddot{y} + 2\dot{x}\Omega - \Omega^2 y = -y\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right)}$$

$$\mathcal{L}_3 = \cancel{m_3}\ddot{z} + G\cancel{m_3}z\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right)$$

$$= \ddot{z} = -Gz\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right)$$

$$\boxed{= \ddot{z} = -z\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right)}$$

$$\left.\begin{aligned}
\ddot{\rho} &= \frac{d\ddot{r}}{dt^2}\cdot\frac{t_c^2}{r_{12}} = \frac{d^2\rho}{d\tau^2} \quad\rightarrow\quad \tau = t/t_c \\
\vec{\rho} &= \frac{\vec{r}}{r_{12}} = x^*\hat{i} + y^*\hat{j} + z^*\hat{k} \\
\vec{\sigma} &= \frac{\vec{r_1}}{r_{12}} = (x^* + \pi_2)\hat{i} + y^*\hat{j} + z^*\hat{k} \\
\vec{\psi} &= \frac{\vec{r_2}}{r_{12}} = (x^* - 1 + \pi_2)\hat{i} + y^*\hat{j} + z^*\hat{k}
\end{aligned}\right\}\ \underline{\text{time scaling}}$$

$$\ddot{x} - 2\dot{y}\Omega - \Omega^2 x = -\frac{\mu_1(x + \pi_2 r_{12})}{r_1^3} - \frac{\mu_2(x - \pi_1 r_{12})}{r_2^3}$$

$$|\psi| = \frac{r_2}{r_{12}} \rightarrow r_2 = \psi r_{12} \rightarrow r_2^3 = \psi^3 r_{12}^3$$

$$|\sigma| = \frac{r_1}{r_{12}} \rightarrow r_1 = \sigma r_{12} \rightarrow r_1^3 = \sigma^3 r_{12}^3$$

$$\ddot{x} - 2\dot{y}\Omega - \Omega^2 x = \frac{1}{r_{12}^2}\left(-\frac{\mu_1(x^* + \pi_2)}{\sigma^3} - \frac{\mu_2(x^* - \pi_1)}{\psi^3}\right)$$

$$= \frac{G(m_1 + m_2)}{r_{12}^2}\left(-\frac{\frac{m_1}{m_1+m_2}(x^* + \pi_2)}{\sigma^3} - \frac{\frac{m_2}{m_1+m_2}(x^* - \pi_1)}{\psi^3}\right)$$

$$= \frac{\mu}{r_{12}^2}\left(-\frac{(1 - \pi_2)(x^* + \pi_2)}{\sigma^3} - \frac{\pi_2(x^* - 1 + \pi_2)}{\psi^3}\right)$$

$$\frac{r_{12}^2}{\mu}\left(\ddot{x} - 2\dot{y}\Omega - \Omega^2 x\right) = \left(-\frac{(1 - \pi_2)(x^* + \pi_2)}{\sigma^3} - \frac{\pi_2(x^* - 1 + \pi_2)}{\psi^3}\right)$$

$$r_{12}^2 \quad \Big(\quad \sigma^3 \qquad \overline{\qquad \psi^3} \quad \Big)$$

$$= \frac{\mu}{r_{12}^2}\left(-\frac{(1-\pi_2)(x^*+\pi_2)}{\sigma^3} - \frac{\pi_2(x^*-1+\pi_2)}{\psi^3}\right)$$

$$\frac{r_{12}^2}{\mu}\left(\ddot{x} - 2\dot{y}\Omega - \Omega^2 x\right) = \left(-\frac{(1-\pi_2)(x^*+\pi_2)}{\sigma^3} - \frac{\pi_2(x^*-1+\pi_2)}{\psi^3}\right)$$

$$\frac{r_{12}^3}{\mu} = t_c^2 = \frac{1}{\Omega^2} \longrightarrow * \text{ 1 time step } \tau = 1 \text{ degree } \theta \; *$$

$$\frac{1}{r_{12}}\left(\frac{\ddot{x}}{\Omega^2} - \frac{2\dot{y}}{\Omega} - x\right) = \dots$$

$$\left.\begin{aligned} \ddot{x} &= \frac{\partial}{\partial t}\frac{\partial x}{\partial t} \\ \dot{y} &= \frac{\partial y}{\partial t} \end{aligned}\right\} \quad \tau = \frac{t}{t_c} = t\Omega \rightarrow \boxed{dt = \frac{1}{\Omega}d\tau}$$

$$\frac{1}{r_{12}}\left(\frac{\partial}{\partial t}\frac{\partial x}{\partial t}\frac{1}{\Omega^2} - 2\frac{\partial y}{\partial t}\frac{1}{\Omega} - x\right) = \dots$$

$$\frac{\partial}{\partial t}\frac{\partial x^*}{\partial t}\frac{1}{\Omega^2} - 2\frac{\partial y^*}{\partial t}\frac{1}{\Omega} - x^* = \dots$$

$$\cancel{\Omega^2}\frac{\partial}{\partial \tau}\frac{\partial x^*}{\partial \tau}\cancel{\frac{1}{\Omega^2}} - 2\frac{\partial y^*}{\partial \tau}\cancel{\frac{1}{\Omega}}\cancel{\Omega} - x^* = \dots$$

$$\boxed{\ddot{x}^* - 2\dot{y}^* - x^* = -\frac{(1-\pi_2)(x^*+\pi_2)}{\sigma^3} - \frac{\pi_2(x^*-1+\pi_2)}{\psi^3}} \rightarrow \text{\& so on...}$$

Linearized equations: everything depends on x, y, z

$\mathcal{L}_{\#}(x, \dot{x}, \ddot{x}, y, \dot{y}, \ddot{y}, z, \dot{z}, \ddot{z})$

$$\ddot{x}^* - 2\dot{y}^* - x^* = \frac{1-\pi_2}{\sigma^3}(x^* + \pi_2) - \frac{\pi_2}{\psi^3}(x^* - 1 + \pi_2)$$

$$\mathcal{L}_1 = x + 2\dot{y} - \ddot{x} + \frac{1-\pi_2}{((x+\pi_2)^2+y^2+z^2)^{3/2}}(x+\pi_2) - \frac{\pi_2}{((x-1+\pi_2)^2+y^2+z^2)^{3/2}}(x-1+\pi_2)$$

$$\mathcal{L}_1 = \mathcal{L}_1\big|_0 + \frac{\partial \mathcal{L}}{\partial x}\Big|_0 \Delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_0 \Delta \dot{x} + \frac{\partial \mathcal{L}}{\partial \ddot{x}}\Big|_0 \Delta \ddot{x} + \frac{\partial \mathcal{L}}{\partial y}\Big|_0 \Delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\Big|_0 \Delta \dot{y} + \frac{\partial \mathcal{L}}{\partial \ddot{y}}\Big|_0 \Delta \ddot{y} + \frac{\partial \mathcal{L}}{\partial z}\Big|_0 \Delta z + \frac{\partial \mathcal{L}}{\partial \dot{z}}\Big|_0 \Delta \dot{z} + \frac{\partial \mathcal{L}}{\partial \ddot{z}}\Big|_0 \Delta \ddot{z}$$

$$= \Big\{1 + (1-\pi_2)\big[-\tfrac{3}{2}\sigma^{-5} x(2)(x+\pi_2) + \sigma^{-3} - (\pi_2)\big[(x-1+\pi_2)(-\tfrac{3}{2})(2)(\psi^{-5}) - \psi^{-3}\big]\big]\Big\}\Delta x - \Delta \ddot{x} +$$

$$+ \Big[(1-\pi_2)(x+\pi_2)(-\tfrac{3}{2})(2)(\sigma^{-5}) - (\pi_2)(x-1+\pi_2)(-\tfrac{3}{2})(2)(\psi^{-5})\Big] y\,\Delta y + 2\Delta\dot{y} +$$

$$+ \Big[(1-\pi_2)(x+\pi_2)(-\tfrac{3}{2})(2)(\sigma^{-5}) - (\pi_2)(x-1+\pi_2)(-\tfrac{3}{2})(2)(\psi^{-5})\Big] z\,\Delta z$$

$$\boxed{\begin{aligned}\mathcal{L}_1 = &\left[1 - 3\left\{(1-\pi_2)\left[\frac{(x+\pi_2)^2}{\sigma^5} + \frac{1}{3\sigma^3}\right] - \pi_2\left[\frac{(x-1+\pi_2)^2}{\psi^5} + \frac{1}{3\psi^3}\right]\right\}\right]\Delta x - \Delta\ddot{x} \\ &- 3\left[\frac{(1-\pi_2)(x+\pi_2)}{\sigma^5} - \frac{\pi_2(x-1+\pi_2)}{\psi^5}\right]y\Delta y + 2\Delta\dot{y} - 3\left[\frac{(1-\pi_2)(x+\pi_2)}{\sigma^5} - \frac{\pi_2(x-1+\pi_2)}{\psi^5}\right]z\Delta z\end{aligned}}$$

$$\ddot{y}^* + 2\dot{x}^* - y^* = \frac{1-\pi_2}{\sigma^3}y^* - \frac{\pi_2}{\psi^3}y^*$$

$$\ddot{y}^* + 2\dot{x}^* - y^* = \frac{1-\pi_2}{\sigma^3} y^* - \frac{\pi_2}{\psi^3} y^*$$

$$\mathcal{L}_2 = y - 2\dot{x} + \ddot{y} + \frac{(1-\pi_2)y}{\sigma^3} - \frac{\pi_2 y}{\psi^3}$$

$$\mathcal{L}_2 = \mathcal{L}_2\big|_0 + \frac{\partial \mathcal{L}}{\partial x}\Big|_0 \Delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_0 \Delta \dot{x} + \frac{\partial \mathcal{L}}{\partial \ddot{x}}\Big|_0 \Delta \ddot{x} + \frac{\partial \mathcal{L}}{\partial y}\Big|_0 \Delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\Big|_0 \Delta \dot{y} + \frac{\partial \mathcal{L}}{\partial \ddot{y}}\Big|_0 \Delta \ddot{y} + \frac{\partial \mathcal{L}}{\partial z}\Big|_0 \Delta z + \frac{\partial \mathcal{L}}{\partial \dot{z}}\Big|_0 \Delta \dot{z} + \frac{\partial \mathcal{L}}{\partial \ddot{z}}\Big|_0 \Delta \ddot{z}$$

$$= \Big[(1-\pi_2)y(-3/2)(2)\sigma^5 - \pi_2 y(-3/2)(2)\psi^5\Big]\Delta x - 2\Delta\dot{x} +$$

$$+\Big[1 + (1-\pi_2)\Big[\frac{1}{\sigma^3} + y(-3/2)(2)\sigma^5\Big] + \pi_2\Big[\frac{1}{\psi^3} + y(-3/2)(2)\psi^5\Big]\Big]\Delta y + \Delta\ddot{y} +$$

$$+\Big[(1-\pi_2)y(-3/2)(2)\sigma^5 - \pi_2 y(-3/2)(2)\psi^5\Big]\Delta z$$

$$(x^2+y^2+z^2)^{-\frac{3}{2}} = -3y(x^2+y^2+z^2)$$

$$\boxed{\mathcal{L}_2 = -3\Big[\frac{(1-\pi_2)y(x+\pi_2)}{\sigma^5} - \frac{\pi_2 y(x-1+\pi_2)}{\psi^5}\Big]\Delta x - 2\Delta\dot{x} + \Big[1 + (1-\pi_2)\Big[\frac{1}{\sigma^3} - \frac{3y^2}{\sigma^5}\Big] + \pi_2\Big[\frac{1}{\psi^3} - \frac{3y^2}{\psi^5}\Big]\Big]\Delta y + \Delta\ddot{y} - 3\Big[\frac{(1-\pi_2)y}{\sigma^5} - \frac{\pi_2 y}{\psi^5}\Big]z\Delta z}$$

$$\ddot{z}^* = -\frac{1-\pi_2}{\sigma^3} z^* - \frac{\pi_2}{\psi^3} z^*$$

$$\mathcal{L}_3 = -\ddot{z} - \frac{(1-\pi_2)}{\sigma^3} z - \frac{\pi_2}{\psi^3} z$$

$$\mathcal{L}_3 = \mathcal{L}_3\big|_0 + \frac{\partial \mathcal{L}}{\partial x}\Big|_0 \Delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_0 \Delta \dot{x} + \frac{\partial \mathcal{L}}{\partial \ddot{x}}\Big|_0 \Delta \ddot{x} + \frac{\partial \mathcal{L}}{\partial y}\Big|_0 \Delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\Big|_0 \Delta \dot{y} + \frac{\partial \mathcal{L}}{\partial \ddot{y}}\Big|_0 \Delta \ddot{y} + \frac{\partial \mathcal{L}}{\partial z}\Big|_0 \Delta z + \frac{\partial \mathcal{L}}{\partial \dot{z}}\Big|_0 \Delta \dot{z} + \frac{\partial \mathcal{L}}{\partial \ddot{z}}\Big|_0 \Delta \ddot{z}$$

$$= \left[(1-\pi_2) y \left(-\tfrac{3}{2}\right)(2) \sigma^5 - \pi_2 y \left(-\tfrac{3}{2}\right)(2) \psi^5 \right] \Delta x - 2\Delta\dot{x} +$$

$$+ \left[1 + (1-\pi_2)\left[\frac{1}{\sigma^3} + y\left(-\tfrac{3}{2}\right)(2)\sigma^5 \right] + \pi_2 \left[\frac{1}{\psi^3} + y\left(-\tfrac{3}{2}\right)(2)\psi^5 \right] \right] \Delta y + \Delta\ddot{y} +$$

$$+ \left[(1-\pi_2) y \left(-\tfrac{3}{2}\right)(2) \sigma^5 - \pi_2 y \left(-\tfrac{3}{2}\right)(2) \psi^5 \right] \Delta z$$

$$\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} = -3y\left(x^2+y^2+z^2\right)$$

$$\boxed{\mathcal{L}_2 = -3\left[\frac{(1-\pi_2) y (x+\pi_2)}{\sigma^5} - \frac{\pi_2 y (x-1+\pi_2)}{\psi^5} \right] \Delta x - 2\Delta\dot{x} + \left[1 + (1-\pi_2)\left[\frac{1}{\sigma^3} - \frac{3y^2}{\sigma^5} \right] + \pi_2\left[\frac{1}{\psi^3} - \frac{3y^2}{\psi^5} \right] \right] \Delta y + \Delta\ddot{y} - 3\left[\frac{(1-\pi_2) y}{\sigma^5} - \frac{\pi_2 y}{\psi^5} \right] z\Delta z}$$

$$\ddot{z}^* = -\frac{1-\pi_2}{\sigma^3} z^* - \frac{\pi_2}{\psi^3} z^*$$

$$\mathcal{L}_3 = -\ddot{z} - \frac{(1-\pi_2)}{\sigma^3} z - \frac{\pi^2}{\psi^3} z$$

$$\mathcal{L}_3 = \mathcal{L}_3\big|_0 + \frac{\partial \mathcal{L}}{\partial x}\Big|_0 \Delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_0 \Delta\dot{x} + \frac{\partial \mathcal{L}}{\partial \ddot{x}}\Big|_0 \Delta\ddot{x} + \frac{\partial \mathcal{L}}{\partial y}\Big|_0 \Delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\Big|_0 \Delta\dot{y} + \frac{\partial \mathcal{L}}{\partial \ddot{y}}\Big|_0 \Delta\ddot{y} + \frac{\partial \mathcal{L}}{\partial z}\Big|_0 \Delta z + \frac{\partial \mathcal{L}}{\partial \dot{z}}\Big|_0 \Delta\dot{z} + \frac{\partial \mathcal{L}}{\partial \ddot{z}}\Big|_0 \Delta\ddot{z}$$

$$\boxed{\mathcal{L}_3 = 3z\left(\frac{(1-\pi_2)(x+\pi_2)}{\sigma^5} + \frac{\pi^2 (x-1+\pi_2)}{\psi^5} \right) \Delta x + 3z\left(\frac{(1-\pi_2)}{\sigma^5} + \frac{\pi^2}{\psi^5} \right) y\Delta y + \left\{ 3z^2\left(\frac{(1-\pi_2)}{\sigma^5} + \frac{\pi^2}{\psi^5} \right) + \frac{(1-\pi_2)}{\sigma^3} + \frac{\pi^2}{\psi^3} \right\} \Delta z - \Delta\ddot{z}}$$

* Subs. Lagrange point coordinates → from Proj 1 ; μ, $1-\mu$

$\Delta x, \Delta y \Rightarrow$ coupled

$\Delta z \Rightarrow$ only of Δz

Using the non-linearized eqns & all velocities & acc $=0$ (def of L.P.)

$$X = \frac{(1-\pi_2)(x+\pi_2)}{\sigma^3} + \frac{\pi_2(x-1+\pi_2)}{\psi^3}$$

$$1 = \left(\frac{(1-\pi_2)}{\sigma^3} + \frac{\pi_2}{\psi^3}\right)$$

$$\downarrow$$

$$0 = \left(\frac{1-\pi_2}{\sigma^3} + \frac{\pi_2}{\psi^3}\right) z$$

try to find LP coordinates

$0 = z$

- $y = 0 \rightarrow L_1, L_2, L_3$
- $y \neq 0 \rightarrow L_4, L_5$

$\rightarrow$ planar: so same coordinates as Proj 1

	x	y	z
L_1	.836915	0	0
L_2	1.15568	0	0
L_3	[illegible]0506	0	0

$\rightarrow$ Earth-Moon

	x	y	z
L_1	.836915	0	0
L_2	1.15568	0	0
L_3	-1.00506	0	0
L_4	$1/2 - \pi_2$	$\sqrt{3}/2$	0
L_5	$1/2 - \pi_2$	$-\sqrt{3}/2$	0

→ Earth-Moon

z is always zero for these eqns, so the linearized eqns reduce to:

$$\mathcal{L}_1 = \left[1 - 3\left\{(1-\pi_2)\left[\frac{(x+\pi_2)^2}{\sigma^5} + \frac{1}{3\sigma^3}\right] - \pi_2\left[\frac{(x-1+\pi_2)^2}{\psi^5} + \frac{1}{3\psi^3}\right]\right\}\right]\Delta x - \Delta\ddot{x}$$

$$-3\left[\frac{(1-\pi_2)(x+\pi_2)}{\sigma^5} - \frac{\pi_2(x-1+\pi_2)}{\psi^5}\right]y\Delta y + 2\Delta\dot{y} \cancel{- 3\left[\frac{(1-\pi_2)(x+\pi_2)}{\sigma^5} - \frac{\pi_2(x-1+\pi_2)}{\psi^5}\right]z\Delta z}$$

$$\boxed{\begin{aligned}\Delta\ddot{x} - 2\Delta\dot{y} = {} & \left[1 - 3\left\{(1-\pi_2)\left[\frac{(x+\pi_2)^2}{\sigma^5} + \frac{1}{3\sigma^3}\right] - \pi_2\left[\frac{(x-1+\pi_2)^2}{\psi^5} + \frac{1}{3\psi^3}\right]\right\}\right]\Delta x - \\ & -3\left[\frac{(1-\pi_2)(x+\pi_2)}{\sigma^5} - \frac{\pi_2(x-1+\pi_2)}{\psi^5}\right]y\Delta y\end{aligned}}$$

$$\mathcal{L}_2 = -3\left[\frac{(1-\pi_2)y(x+\pi_0)}{\sigma^5} - \frac{\pi_2 y(x-1+\pi_0)}{\psi^5}\right]\Delta x - 2\Delta\dot{x} + \left[1 + (1-\pi_0)\left[\frac{1}{\sigma^3} - \frac{3y^2}{\sigma^5}\right] + \pi_0\left[\frac{1}{\psi^3} - \frac{3y^2}{\psi^5}\right]\right]\Delta y + \Delta\ddot{y} -$$

$$\cancel{-3\left[\frac{(1-\pi_2)y}{\sigma^5} - \frac{\pi_2 y}{\psi^5}\right]z\Delta z}$$

$$\boxed{\Delta\ddot{y} + 2\Delta\dot{x} = \left[1 + (1-\pi_0)\left[\frac{1}{\sigma^3} - \frac{3y^2}{\sigma^5}\right] + \pi_0\left[\frac{1}{\psi^3} - \frac{3y^2}{\psi^5}\right]\right]\Delta y - 3\left[\frac{(1-\pi_2)y(x+\pi_0)}{\sigma^5} - \frac{\pi_2 y(x-1+\pi_0)}{\psi^5}\right]\Delta x}$$

$$\mathcal{L}_3 = \cancel{3z\left(\frac{(1-\pi_2)(x+\pi_0)}{\sigma^5} + \frac{\pi^2(x-1+\pi_0)}{\psi^5}\right)\Delta x} + \cancel{3z\left(\frac{(1-\pi_2)}{\sigma^5} + \frac{\pi^2}{\psi^5}\right)y\Delta y} +$$

$$+ \left[\cancel{3z^2\left(\frac{(1-\pi_2)}{\sigma^5} + \frac{\pi^2}{\psi^5}\right)} + \frac{(1-\pi_2)}{\sigma^3} + \frac{\pi^2}{\psi^3}\right]\Delta z - \Delta\ddot{z}$$

$$\boxed{\Delta\ddot{z} = \left[\frac{(1-\pi_2)}{\sigma^3} + \frac{\pi^2}{\psi^3}\right]\Delta z}$$

$$
\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta \dot{x} \\ \Delta \dot{y} \\ \Delta \dot{z} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \left[1-3\left[(1-\pi_2)\left[\frac{(\lambda+\pi_2)^2}{\sigma^5}+\frac{1}{3\sigma^3}\right]-\pi_2\left[\frac{(\lambda-1+\pi_2)^2}{\psi^5}+\frac{1}{3\psi^3}\right]\right]\right]x_1 - 3\left[\frac{(1-\pi_2)(\lambda+\pi_2)}{\sigma^5}-\frac{\pi_2(\lambda-1+\pi_2)}{\psi^5}\right]y\,x_2 + 2x_5 \\ \left[1+(1-\pi_2)\left[\frac{1}{\sigma^3}-\frac{3y^2}{\sigma^5}\right]+\pi_2\left[\frac{1}{\psi^3}-\frac{3y^2}{\psi^5}\right]\right]x_2 - 3\left[\frac{(1-\pi_2)(\lambda+\pi_2)}{\sigma^5}-\frac{\pi_2(\lambda-1+\pi_2)}{\psi^3}\right]y\,x_1 - 2x_4 \\ \left[\frac{(1-\pi_2)}{\sigma^3}+\frac{\pi_2}{\psi^3}\right]x_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \left[1-3\left[(1-\pi_2)\left[\frac{(\lambda+\pi_2)^2}{\sigma^5}+\frac{1}{3\sigma^3}\right]-\pi_2\left[\frac{(\lambda-1+\pi_2)^2}{\psi^5}+\frac{1}{3\psi^3}\right]\right]\right] & -3\left[\frac{(1-\pi_2)(\lambda+\pi_2)}{\sigma^5}-\frac{\pi_2(\lambda-1+\pi_2)}{\psi^5}\right]y & 0 & 0 & 2 & 0 \\ -3\left[\frac{(1-\pi_2)(\lambda+\pi_2)}{\sigma^5}-\frac{\pi_2(\lambda-1+\pi_2)}{\psi^3}\right]y & \left[1+(1-\pi_2)\left[\frac{1}{\sigma^3}-\frac{3y^2}{\sigma^5}\right]+\pi_2\left[\frac{1}{\psi^3}-\frac{3y^2}{\psi^5}\right]\right] & 0 & -2 & 0 & 0 \\ 0 & 0 & \left[\frac{(1-\pi_2)}{\sigma^3}+\frac{\pi_2}{\psi^3}\right] & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}
$$

$$\ddot{z} = -\left(\frac{1-\pi}{\sigma^3} + \frac{\pi}{\psi^3}\right) z$$

$$\ddot{z} + \left(\frac{1-\pi}{\sigma^3} + \frac{\pi}{\psi^3}\right) z = 0$$

$$\boxed{\ddot{x} + kx = 0}$$

$$x(t) = A\sin(\omega t + \phi)$$

$$w^2 = k$$

$$\left.\begin{aligned} x_0 &= A_0 \sin\phi \\ \dot{x}_0 &= A_0 w \cos\phi \end{aligned}\right] \rightarrow \tan\phi = \frac{w x_0}{\dot{x}_0}$$

$$W_z^2 = \frac{1-\pi}{\sigma^3} + \frac{\pi}{\psi^3}$$

$$W_z = \left(\frac{1-\pi}{\sigma^3} + \frac{\pi}{\psi^3}\right)^{1/2}$$

$$z(t) = A_z \sin(w_z t + \phi)$$

$$\phi = \tan^{-1}\left(\frac{w_z z_0}{\dot{z}_0}\right)$$

$x + \pi$

$x - 1 + \pi$

for L4:

$$\left.\begin{aligned} x &= 1/2 - \pi_2 \\ y &= \sqrt{3}/2 \\ z &= 0 \end{aligned}\right\} \quad \pi_2 = 1.2151 \times 10^{-2} \quad \left\}\begin{aligned} x &= .487849 \\ y &= .86605 \\ z &= 0 \end{aligned}\right\} \quad \begin{aligned} \sigma &= 1 \\ \psi &= 1 \end{aligned}$$

$$W = \left(1 - \pi + \pi\right)^{1/2} = 1$$

$$Z(t) = A_z \sin\left(t + \phi\right)$$

Amplitude, slightly larger than R_{moon}

→ Scaled

$$A_z = .004551694687 = \frac{R_{moon}}{R_{E \to moon}}$$

$$\phi = 30^\circ$$

$$\dot{Z}(t) = A_z W \cos\phi$$

for L_2:

$$\left.\begin{aligned} X &= 1.15568 \\ y &= 0 \\ z &= 0 \\ \pi &= 1.2151 \cdot 10^{-2} \end{aligned}\right\} \quad \left.\begin{aligned} \sigma &= 1.167831 \\ \psi &= .167831 \end{aligned}\right\} \quad W_z = \left(\frac{1-\pi}{\sigma^3} + \frac{\pi}{\psi^3}\right)^{1/2} = \left(\frac{.9999878}{(1.167831)^3} + \frac{1.2151 \cdot 10^{-2}}{(.167831)^3}\right)^{1/2}$$

$$= 1.78836$$

Results:

In this analysis, the system in question was the Earth-Moon system. The first Lagrange Point (LP) that was analyzed was LP2. This point is located at the far side of Moon.

The state-space of the linearized equations of motion was used to find the stability of the system. The eigenvalues of the six-by-six matrix were taken and analyzed. It was found that LP2 was unstable, having positive real roots.

This analysis solved differential equations numerically, using ODE 45 to find position and velocity of the satellite.

The position of a satellite in the neighborhood of LP2 in the b frame is shown in Figure 1.

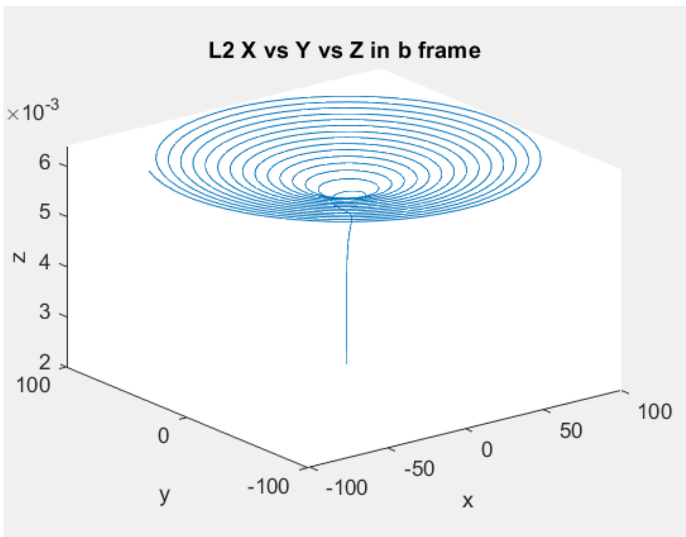


Figure 1

This orbit visualizes the halo orbit, spinning over and over. Changing the x position initial condition allowed for the change in the amplitude of the z change. Figure 2 shows a larger x initial condition.

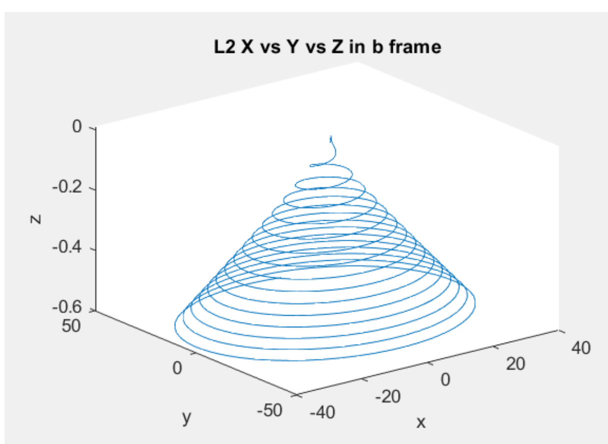


Figure 2

The z coordinate moves further into the negative direction. It also had a smaller range in which x and y span out to over the time period. Messing around with the initial conditions more could result in a spiral that remains more stable and doesn't expand outward.

Figure 3 shows how z changes over time for the initial conditions imposed for Figure 1.

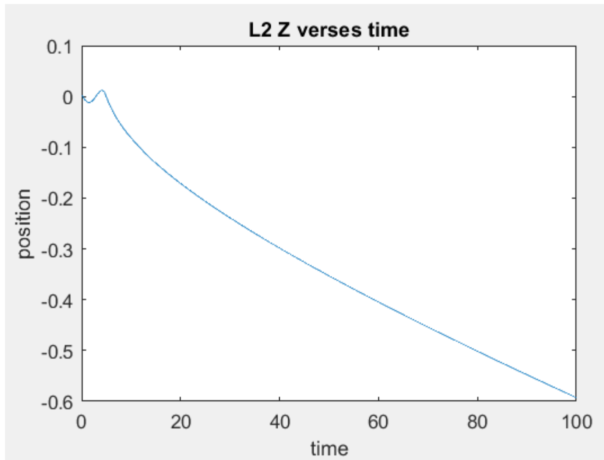


Figure 3

Again, changing the initial conditions would change the magnitude in which z would change over time.

Finding an analytical solution, using the linearized state-space matrix and solving it using the eigenvalues and eigenvectors, did not show promising results. These solutions used a perturbation, which was the same values used for Project 1.

Figure 4 is the x versus y graph of the perturbed analytical solution.

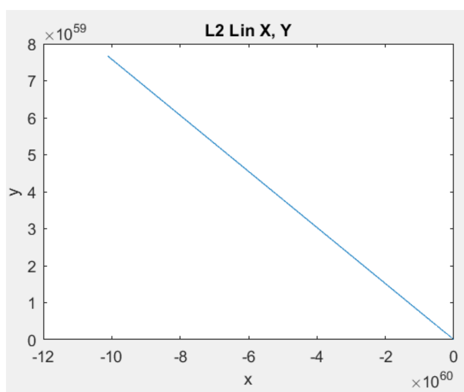


Figure 4

The solution goes off to astronomical numbers. This is due to the fact that LP2 is unstable, thus, the analytical solution does not exist.

The second Lagrange Point that was analyzed with LP4. Unlike LP2, it was found that LP4 was stable. From considering the unstable results of LP2, it was predicted that a satellite in the neighborhood of LP4 will have an analytical solution that exists.

The x and y position of a satellite in the neighborhood of LP4 in the b frame is shown in Figure 5.

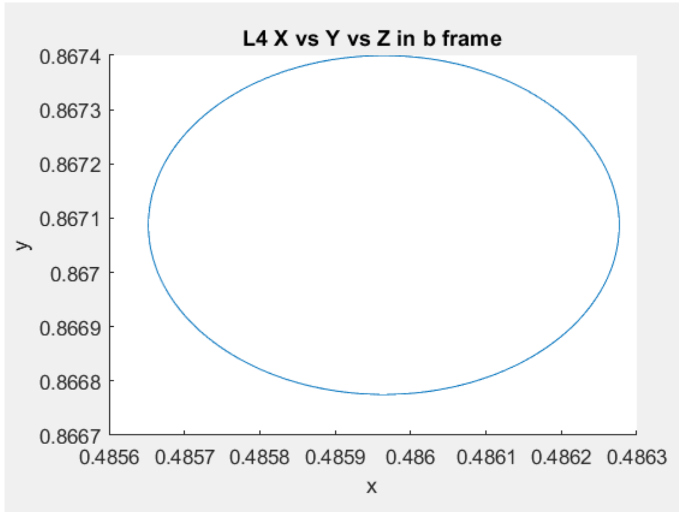


Figure 5

The x versus z graph of LP4 is shown in Figure 6.

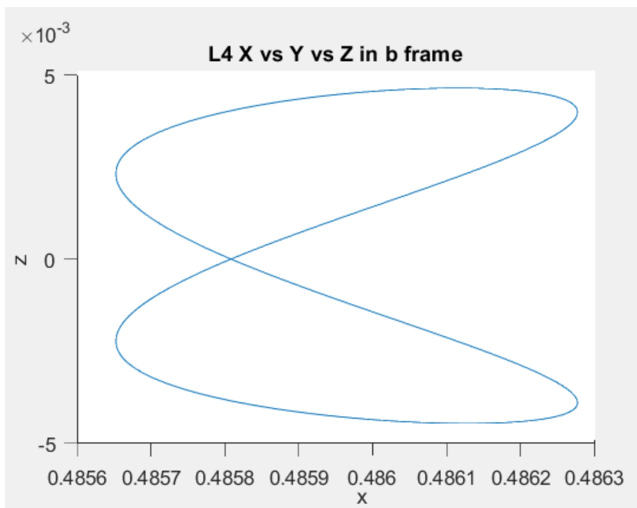


Figure 6

This orbit loops around in a figure 8 pattern throughout the duration of the orbit. Changing the x and y initial conditions did not change very much of the graph.

However, z does changes on a very small scale throughout the simulation, as shown in Figure 7.

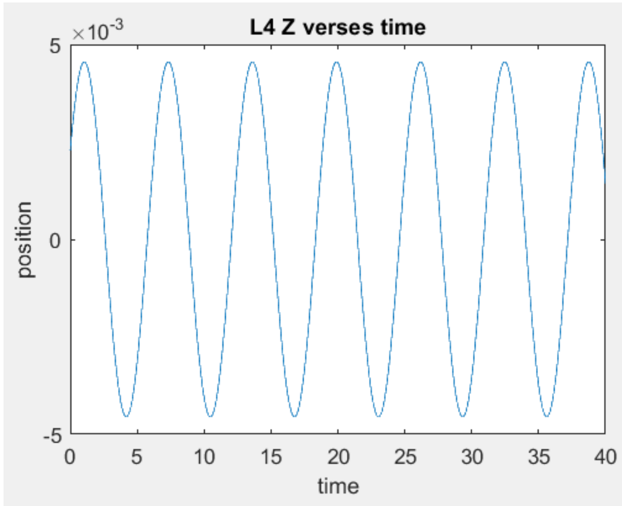


Figure 7

This shows the frequency that was found by solving the nonlinear z equation of motion. The small scale explains why the orbit remains extremely stable throughout the duration of the simulation. It is also important to notice how this movement is decoupled from x and y, which explains why it does not change over time.

The change in the x and y position compared to the initial conditions was also visualized in figure 8.

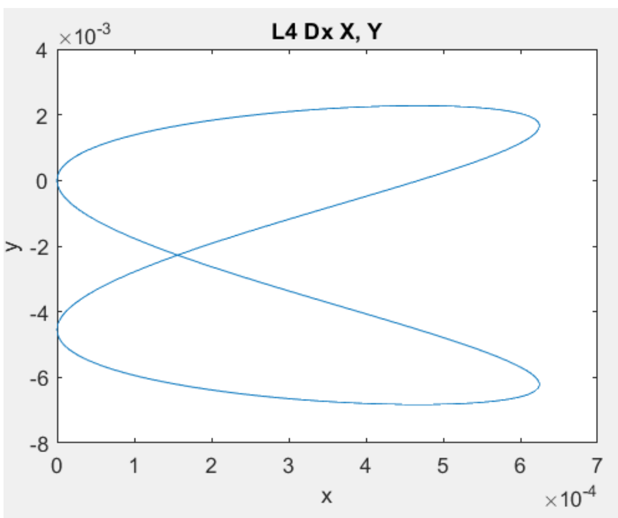


Figure 8

This was found by subtracting the initial conditions from the positions found via ODE45 over the simulation duration. It shows that the changes in both x and y are very small, on the order of 10^-3 to 10^-4.

An analytical solution was also found, using the perturbed condition from project 1. The x versus y graph can be seen in Figure 9.

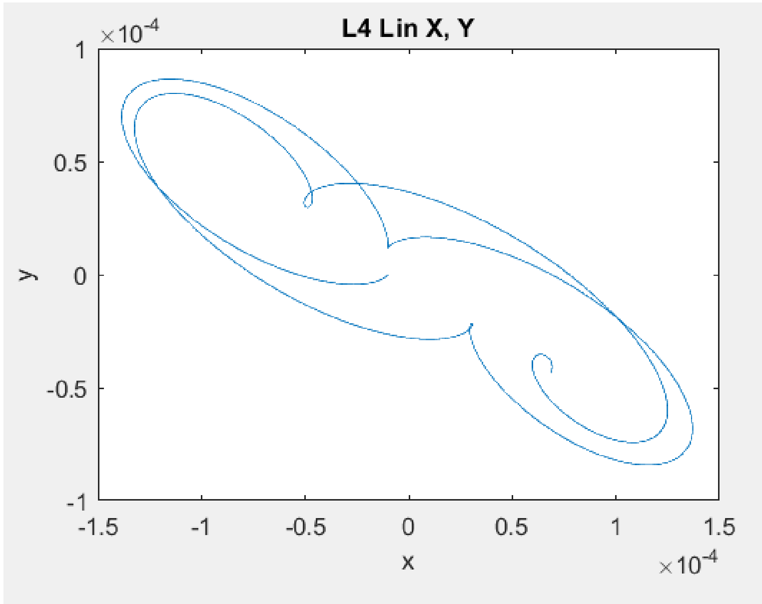


Figure 9

The perturbed position is moved off of the figure 8 pattern, showing an orbit very similar to the LP4 position from Project 1.

The x, y, and z positions over time can be seen in Figures 10 and 11.

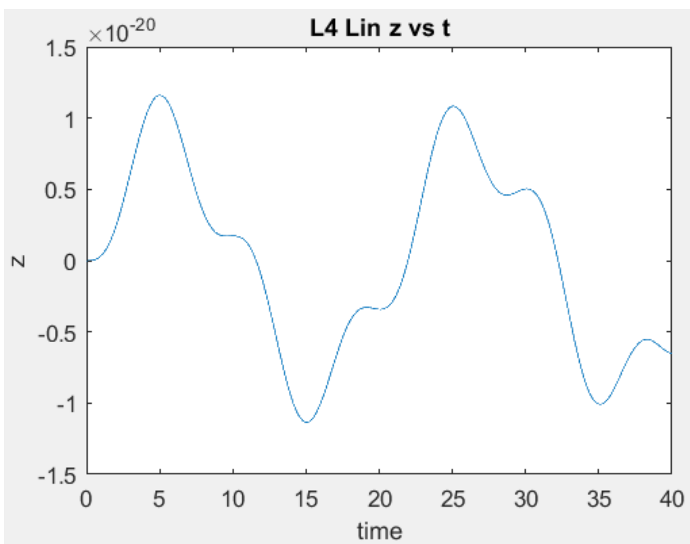


Figure 10

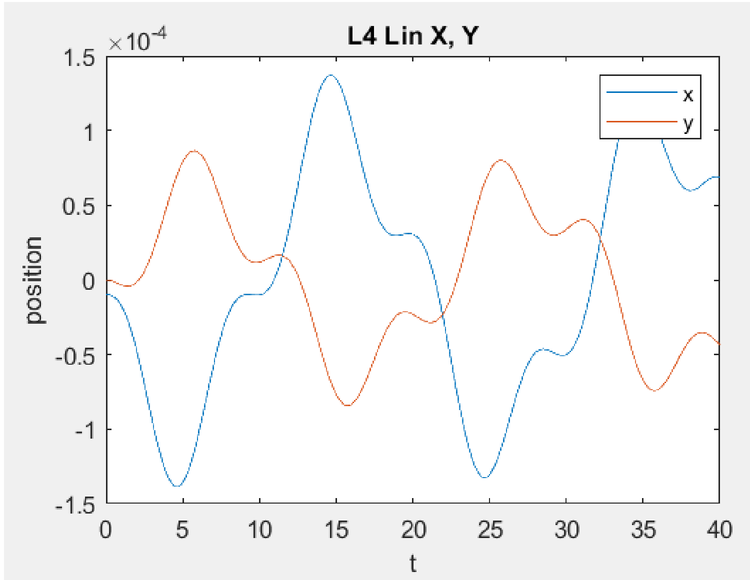


Figure 11

The amplitude of the position change of x and y are both on the same scale, still small, while z is on am even smaller scale. This again shows the decoupling of the equations of motion and how stable the LP4 orbit is.