AERSP 301 Aerospace Structures Extra Credit Project: Beam Bending

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This experiment was set up to compare experimental data with different analytical methods of finding deformations of a bent beam. Three methods were used to determine the deformations at different loads, which included Euler-Bernoulli Beam Bending equations, Finite Element Analysis, and SolidWorks simulations. Comparing data revealed that the beam under analysis had undergone hysteresis and had a non-loaded deformation of .7 cm. Adjusted data then revealed that Euler-Bernoulli and FEA break down at large deformations. Thus, SolidWorks had the best model of the deformations.

I. Nomenclature

σ	=	stress
M_x or u"	=	moment in the x direction
M_y or v"	=	moment in the y direction
I_{xx}	=	second moment of area with respect to the x-axis
I_{yy}	=	second moment of area with respect to the y-axis
I_{xy}	=	product of inertia
w_y	=	distributed load in the y direction
w_x	=	distributed load in the x direction
S_y	=	force in the y direction
S_x	=	force in the x direction
z	=	direction parallel to the length of the beam
u'	=	angle of beam in the x direction
v'	=	angle of beam in the y direction
u	=	displacement in the x direction
v	=	displacement in the y direction
C	=	constant of integration
$\vec{F}$	=	force vector
K_s	=	stiffness matrix
$\vec{q}$	=	displacement and angle vector
w_N	=	displacement
θ_N	=	angle

II. Introduction

This experiment was conducted with the purpose of testing the accuracy of beam-bending principles and simulation methods when compared to experimental data.

The Aerospace Structures course several different methods of predicting deformation of a beam that is subjected to an external force. The first method covered was in relation to Euler-Bernoulli beam theory. This theory considers direct stress at a point and models a deflection relationship by considering loading, load placement, and geometry of the beam. Euler-Bernoulli assumes that plane sections perpendicular to the mid-plane remain plane and perpendicular to the beam axis after deformation. It also assumes that the object under analysis is a long, slender beam. The second method covered was Finite Element Analysis (FEA). This method breaks a beam up into an N number of sections for analysis. This theory considers energy methods, both strain energy and external work, to model displacement and angle

*Third-Year Aerospace Engineering Undergraduate of a beam's Nth element. The third method of data collection in this experiment was the SolidWorks computer software. SolidWorks uses FEA methods to simulate the behavior of objects and assemblies, considering loads applied.

With three varying methods of collecting deflection data, this experiment was set up to analyze the difference in data collected from the theories compared to experimental data. This will reveal the accuracy of the three beam-bending theories.

III. Objectives

The objectives of this experiment were:

- Determine the deflection of a beam under loading using an experimental set-up
- Determine the deflection of the same beam under corresponding loading using the Euler-Bernoulli beam bending method
- Determine the deflection of the same beam under corresponding loading using Finite Element Analysis
- Determine the deflection of the same beam under corresponding loading using SolidWorks' Simulation Software
- Analyze the differences in data between the four methods of collection
- Determine the accuracy of the data and assumptions of each method

IV. Experimental Set Up

This experiment used the following materials:

- A Mayes 30 cm ruler, model number 10761, stainless steel
- A clamp
- A meter-length ruler (meter stick)
- A table
- A plastic cup
- String
- 85 quarters
- Duct Tape
- Scale

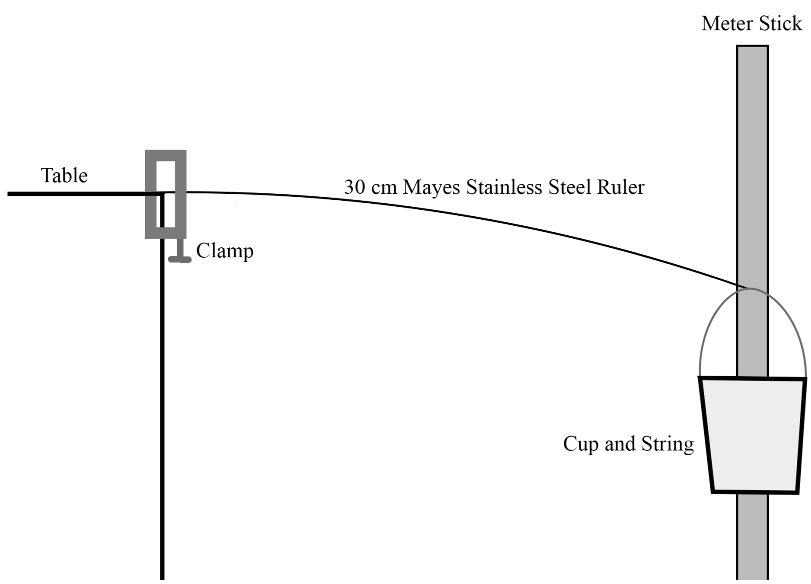

Fig. 1 Experimental Setup

The 30 cm ruler was used as the beam under analysis. This ruler was manufactured to have a hole drilled into the rounded end. About 3 cm of the opposite end of the ruler, the side with a flat edge, was laid onto a table and clamped down. This created a 30.3 cm length between the edge of the camp to the hole. The meter stick was then placed near the free end of the 30cm ruler, such that the hole's location could be easily measured by the meter stick. The meter stick was placed in such a manner that it and the 30 cm ruler almost touched, to ensure good data collection.

The height of the 30 cm ruler's free end was recorded in cm using the digit it was nearest to on the meter stick. This would be the starting deflection, initialized as 0 in later calculations.

The plastic cup was then measured on a scale and the weight was recorded in grams. Holes were then drilled into two opposing sides of the plastic cup, near the top. String was then looped through these holes and through the hole of the ruler. The string was then tied. Duct tape was placed onto all three holes to ensure their stability. The added weight caused deflection to the ruler, so the height of the 30 cm ruler was recorded again.

The 85 quarters were then split up into 17 piles, with 5 quarters each. One quarter was placed on the scale and the weight was recorded.

One pile of 5 quarters was added to the cup and the height of the 30 cm ruler's end was recorded. Another 5 quarters was added, and the height was recorded again. This repeated until the cup held 85 quarters.

The data was then inserted into Excel. The weight of the quarters were converted into Newtons. The deflection of the beam was then calculated, using the first height recording as 0.

V. Modeling

Euler-Bernoulli Methods

Euler-Bernoulli Methods consider direct stress at a point and models a deflection relationship due to pure bending. Euler-Bernoulli also assumes that plane sections perpendicular to the mid-plane remain plane and perpendicular to the beam axis after deformation. It also assumes that the object under analysis is a long, slender beam.

The direct stress of a beam in pure bending can be calculated using

$$\sigma = \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} x + \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} x \tag{1}$$

A relationship between distributed loads, point loads, and moments can also be made through

$$w_y = \frac{-\partial S_y}{\partial z} = \frac{-\partial^2 M_x}{\partial z^2} \tag{2}$$

and

$$w_x = \frac{-\partial S_x}{\partial z} = \frac{-\partial^2 M_y}{\partial z^2} \tag{3}$$

Thus, a relationship can be made that

$$\begin{pmatrix} u" \\ v" \end{pmatrix} = \frac{-1}{E(I_{xx} I_{yy} - I_{xy}^2)} \begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix} \begin{pmatrix} M_x \\ M_y \end{pmatrix} \tag{4}$$

The equations for the secondary moment of areas for a symmetric, square cross-section are

$$I_{xx} = \frac{bh^3}{12} \tag{5}$$

$$I_{yy} = \frac{hb^3}{12} \tag{6}$$

$$I_{xy} = 0 \tag{7}$$

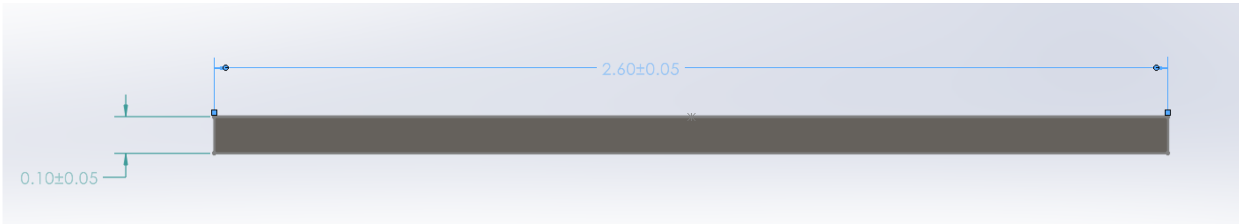

Fig. 2 Cross Section

As per figure 2, the beam has a rectangular cross-section with a width of 2.6 cm and a height of .1 cm. Thus, the secondary moment of areas are calculated to be

$$I_{xx} = \frac{(2.6)(.1)^3}{12} = .000126667\ cm^4 \tag{8}$$

$$I_{yy} = \frac{(.1)(2.)^3}{12} = .14646667\ cm^4 \tag{9}$$

The maximum deformation of the beam under analysis occurred at the maximum loading. Maximum loading for the experiment was a collective weight of the plastic cup and 85 quarters, which comes to 4.8118 Newtons (N) in the negative y direction. At 4.8118 N, using equation 2, the corresponding moment is

$$\frac{-\partial(4.8118)}{\partial z} = \frac{-\partial^2 M_x}{\partial z^2} \tag{10}$$

$$-4.8118 = \frac{-\partial M_x}{\partial z} \tag{11}$$

$$M_x = -4.8118z + C \tag{12}$$

and with the boundary condition of

$$M_x(30.3) = 0 \tag{13}$$

the equation of the moment becomes

$$M_x(30.3) = 0 = -4.8118(30.3) + C \tag{14}$$

$$C = 4.8118(30.3) \tag{15}$$

$$M_x = -4.8118(z - 30.3) \tag{16}$$

Because no forces were present in the x direction, the moment in the y direction is

$$M_y = 0 \tag{17}$$

Thus, using equations 2, 4, 5, and 6, the deformations can be calculated using the reduced equation

$$\begin{pmatrix} u" \\ v" \end{pmatrix} = \frac{-1}{E(I_{xx}I_{yy}} \begin{bmatrix} 0 & I_{xx} \\ I_{yy} & 0 \end{bmatrix} \begin{pmatrix} M_x \\ 0 \end{pmatrix} \tag{18}$$

which reduces to

$$v" = \frac{-1}{EI_{xx}} M_x = \frac{4.8118}{EI_{xx}}(z - 30.3) \tag{19}$$

Since v" corresponds with the moment, then v’ is slope and v is displacement. To find displacement, two derivatives must be taken:

$$v" = \frac{4.8118}{EI_{xx}}(z - 30.3) \tag{20}$$

$$v' = \frac{4.8118}{EI_{xx}}(\frac{z^2}{2} - 30.3z) + C_1 \tag{21}$$

$$v = \frac{4.8118}{EI_{xx}}(\frac{z^3}{3} - \frac{30.3z^2}{2}) + C_1 z + C_2 \tag{22}$$

But with the boundary conditions

$$v(0) = 0 \tag{23}$$

$$v'(0) = 0 \tag{24}$$

then

$$C_1 = 0 \tag{25}$$

$$C_2 = 0 \tag{26}$$

Thus,

$$v = \frac{4.8118}{EI_{xx}}(\frac{z^3}{3} - \frac{30.3z^2}{2}) \tag{27}$$

With the material properties of

$$I_{xx} = .00021667\ cm^4 \tag{28}$$

$$E = 19,000,000\ \frac{N}{cm^2} \tag{29}$$

then the displacement at the end of the beam at maximum load is

$$v = \frac{4.8118\ N}{19,000,000\ \frac{N}{cm^2} * .00021667\ cm^4}(\frac{(30.3)^3}{3} - \frac{30.3(30.3)^2}{2})\ cm^3 \tag{30}$$

$$= -10.8383\ cm \tag{31}$$

Excel was used to calculate the displacement at each loading point to plot the displacement versus force applied.

Finite Element Analysis

Finite Element Analysis (FEA) breaks a beam up into an N number of sections for analysis. This theory considers energy methods, both strain energy and external work, to model displacement and angle of a beam's Nth element. This method allows for a more variety of models, which can include springs, rollers, and torsional springs.

FEA uses an N number of elements to determine angle and displacement of the beam. It does this by assigning an element a displacement and angle at each end of an element of length l.

The equation to find the angles and deformation is

$$\vec{F} = [K_s]\vec{q} \tag{32}$$

Where F is the force vector, K is the stiffness vector, and q is the displacement/angle vector.

When under beam bending, with the force in the y direction, the stiffness matrix of each element is

$$K_s = \frac{2EI_{xx}}{l^3}\begin{bmatrix} 6 & -3l & -6 & -31 \\ -3l & 2l^2 & 3l & l^2 \\ -6 & 3l & 6 & 3l \\ -3l & l^2 & 3l & 2l^2 \end{bmatrix} \tag{33}$$

The experimental beam's dimensions matrix, using two elements, is

$$\begin{matrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{matrix} \tag{34}$$

Thus, the beam's stiffness matrix is

$$K_s = \begin{bmatrix} 14.2066 & -107.147 & -14.2066 & -107.6147 & 0 & 0 \\ -107.6147 & 1086.9 & 107.6147 & 543.4544 & 0 & 0 \\ -14.2066 & 107.6147 & 28.4131 & 0 & -14.2066 & -107.6147 \\ -107.6147 & 543.4544 & 0 & 2173.8 & 107.6147 & 543.4544 \\ 0 & 0 & -14.2066 & 107.6147 & 14.2066 & 107.6147 \\ 0 & 0 & -107.6147 & 543.4544 & 107.6147 & 1086.9 \end{bmatrix} \tag{35}$$

with a corresponding force vector of

$$F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -4.8118 \\ 0 \end{bmatrix} \tag{36}$$

Thus, using the boundary conditions

$$w_1 = 0 \tag{37}$$

$$\theta_1 = 0 \tag{38}$$

Solving for q gets

$$\begin{bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 28.4131 & 0 & -14.2066 & -107.6147 \\ 0 & 2173.8 & 107.6147 & 543.4544 \\ -14.2066 & 107.6147 & 14.2066 & 107.6147 \\ -107.6147 & 543.4544 & 107.6147 & 1086.9 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -4.8118 \\ 0 \end{bmatrix} \tag{39}$$

And with w_3 being the deformation at the end of the beam, then the deformation is

$$w_3 = (2.25247957249302)(-4.8118) = 10.8385\ cm \tag{40}$$

MATLAB was used to calculate displacement per load and to calculate displacement at maximum load per number of elements used. The code was manipulated such that it would run FEA through a loop that calculated displacements for an N number of elements, where N was 1 through 10. The final displacement at max load was consistently 10.8385 cm, so it can be assumed that the displacement per number of elements converged right away. Thus, three elements were used to run the FEA numbers, out of personal preference, since the number of elements used did not have an effect on the data.

SolidWorks

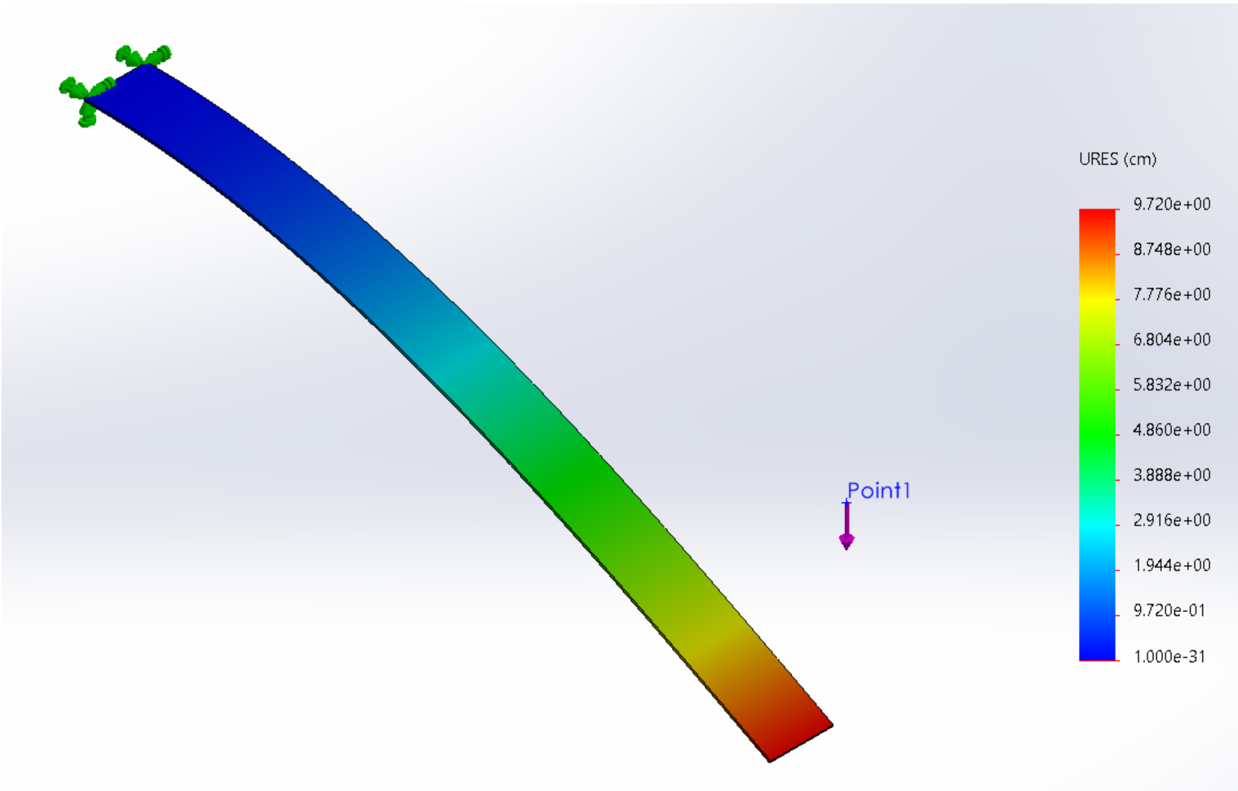

Fig. 3 SolidWorks Simulation at Maximum Loading

SolidWorks uses FEA methods to simulate the behavior of objects and assemblies, considering loads applied.

The beam under analysis was recreated in SolidWorks using exact dimensions. A simulation was set up, under the Simulations tab, to recreate the boundary conditions and the forces.

The ruler used in the experiment was a discontinued product, so information about the type of material it is made of is limited. However, the ruler has "stainless steel" written on it. In addition, SolidWorks does not have an exact material type for stainless steel.

Thus, AISI 304 Steel was used for the SolidWorks simulation. The material properties of 304 are very similar to stainless steel. In addition, Young's Modulus is the only material property used in both Euler-Bernoulli and in FEA. Since 304 and stainless steel have the same Young's Modulus, it can be assumed that using 304 in the simulation will not deem large differences in results.

A fixed boundary condition was applied to one end of the beam using the Fixtures tab. A point load was applied to the opposite end using the External Loads tab.

The simulation was then run. Displacement of the entire beam can be visualized under the Displacement tab, but only the displacement at the loaded end was recorded. In the simulation, this displacement is visualized as the deepest red color.

VI. Results

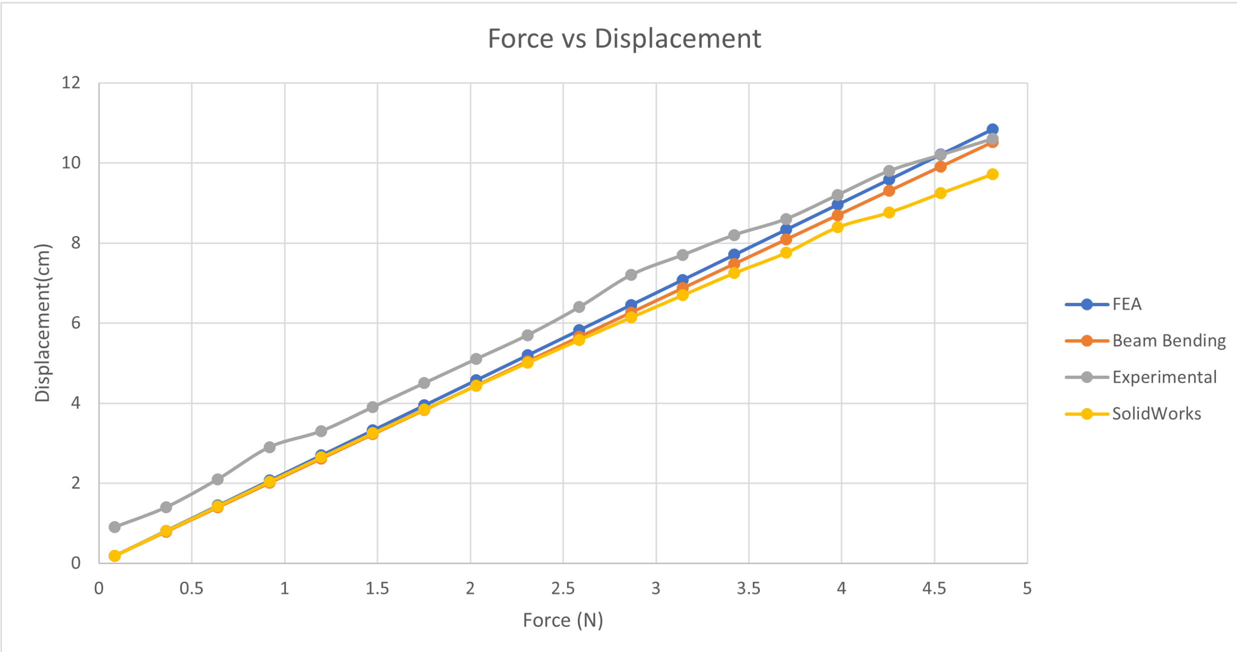

Fig. 4 Force vs Displacement for the Four Methods

The predicted displacement for each added load for each of the three analytical methods are consistent and close to the experimental results, as seen in Figure 4. The experimental displacement tends to be slightly larger than all of the analytical displacements throughout the added loads, but merges near the final load, which can be further confirmed in Table 1.

Table 1 Percent Difference Between Experimental and Analytical Results

Euler-Bernoulli	FEA	SolidWorks
-79.61943634	-79.01111111	-79.25555556
-43.47022271	-41.75714286	-42.48571429
-33.36149089	-31.34285714	-32.28571429
-30.77929309	-28.67931034	-29.79310345
-20.74568759	-18.34242424	-19.84848485
-17.34912516	-14.84102564	-16.71794872
-14.85345441	-12.27555556	-14.55555556
-12.94928705	-10.31372549	-13.07843137
-11.44599702	-8.763157895	-12.05263158
-11.63171908	-8.9546875	-12.78125
-13.00608628	-10.36944444	-14.69444444
-10.75904419	-8.054545455	-13.03896104
-8.786031629	-6.020731707	-11.58536585
-5.956327453	-3.108139535	-9.813953488
-5.481003824	-2.617391304	-8.770652174
-5.063883088	-2.187755102	-10.57142857
-2.826164054	0.117647059	-9.362745098
-0.757329475	2.25	-8.301886792

One possible explanation for this phenomenon is hysteresis. This phenomenon is the explanation as to why material properties change over time [1]. The ruler used for the experiment was old, even being out of production and very hard to find online. It was also found in a garage where it was likely used frequently. That said, the consistent higher displacement of the experimental results could attest to hysteresis. Since the ruler was used frequently, it likely was slightly bend at zero load, so the experimental 0 load did not produce 0 displacement. Considering the data in Table 2, it is likely that the actual displacement at zero load was around .7 cm.

Table 2 Difference Between Experimental and Analytical Results in cm

Euler-Bernoulli	FEA	SolidWorks
0.716574927	0.7111	0.7133
0.608583118	0.5846	0.5948
0.700591309	0.6582	0.678
0.8925995	0.8317	0.864
0.68460769	0.6053	0.655
0.676615881	0.5788	0.652
0.668405449	0.5524	0.655
0.660413639	0.526	0.667
0.65242183	0.4995	0.687
0.744430021	0.5731	0.818
0.936438212	0.7466	1.058
0.828446403	0.6202	1.004
0.720454594	0.4937	0.95
0.512244161	0.2673	0.844
0.504252352	0.2408	0.8069
0.496260543	0.2144	1.036
0.288268733	-0.012	0.955
0.080276924	-0.2385	0.88

An interesting observation of the analytical displacements is that all three methods predict an initial, non-zero displacement within .01 cm. As the load increases, they diverge, following a slightly different slope.

The Beam Bending and FEA analysis produce consistent results, both quite close to each other, following a straight slope. However, SolidWorks tends to break down around a 3.5 N load. This correlates with an error message that SolidWorks displayed when loads past 3.5 N were applied. SolidWorks uses linear theory, which assumes small displacements compared to the length of the beam [2]. The final experimental deformation was 10.6 cm, with a beam length of 30.3 cm. With a 1:3 displacement to length ratio, this falls into a 'large displacement' range. Thus, once the displacement got large enough, the accuracy of the results tend to break down. This explains the large breakdown in results in the larger loading section.

In addition, Euler-Bernoulli assumes small deformations as well. However, a breakdown of results does not occur. As shown in Table 2, the difference between this method and experimental data is larger than the other two methods, but still did not have a significant difference. It also predicted the closest final displacement for the largest load.

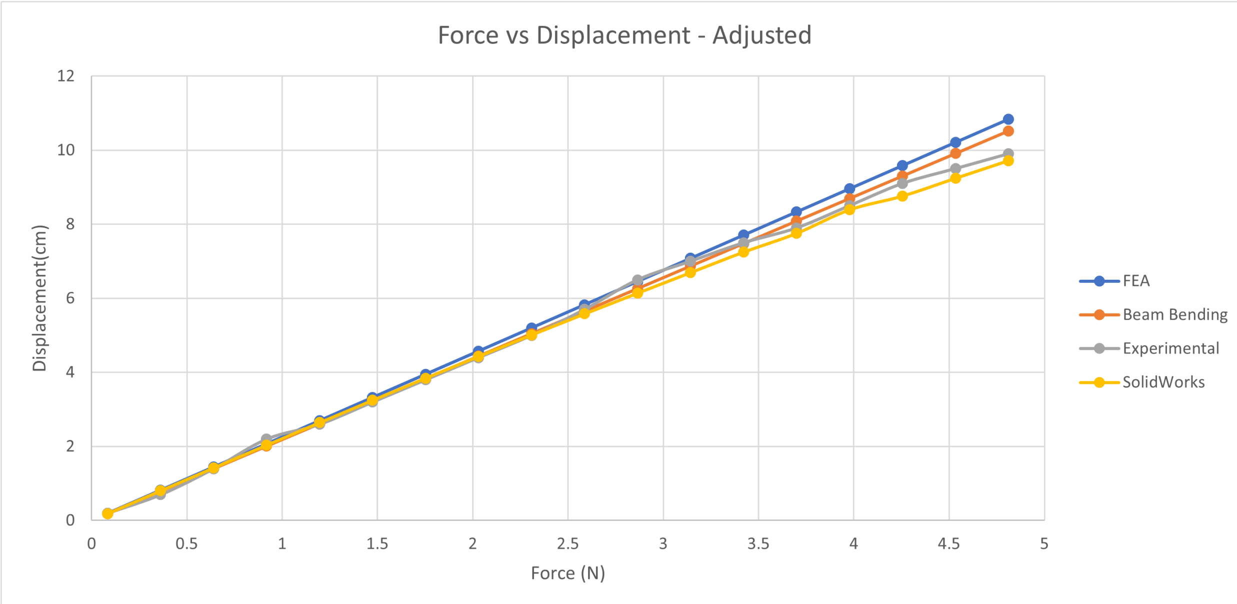

Fig. 5 Force vs Displacement for the Four Methods with Adjusted Experimental Data

An explanation as to why Euler-Bernoulli did not result in skewed data, but SolidWorks did, is likely hysteresis. As mentioned before, an inaccurate zero-force displacement of the ruler most likely screwed the displacement results. Thus, assuming an average difference in data for the first few loadings using Table 2, it can be assumed that the ruler has a .7 cm displacement at zero-loading. Thus, adjusting the data to reflect this theory by creating Figure 5, the breakdown of calculated theories is more easily visualized. It also reflects that SolidWorks is a more accurate model.

VII. Conclusion

In conclusion, this beam-bending experiment compared experimental results with three different analytical results. It was found that the experimental deformation were slightly larger than all three analytical deformations. It can be concluded that this is due to hysteresis. It was also found that analytical solutions broke down after around 3.5 Newtons of force. This is due to the large displacements that the beam underwent during experimentation. The final deformation was one-third of the beam length. This is due to the assumption that beams under analysis of these methods are undergoing small deformations compared to the length. However, the deformations were similar to that of the experiment. Adjusting the data to adhere to the assumption that the beam had deformation when unloaded, the new data shows that SolidWorks is the most accurate simulation. The new data also showed an accurate representation of how Euler-Bernoulli breaks down at large deformations.

References

[1] https://www.lassp.cornell.edu/sethna/hysteresis/WhatIsHysteresis.html

[2] http://help.solidworks.com/2021/english/SolidWorks/cworks/c_Large_Displacement_Solution.html

VIII. Appendix

Raw Data:

	FEA	Euler-B	SW	Experim
Force (N)	Disp (cm)	Disp (cm)	Disp (cm)	Disp (cm)
0.0839	0.1889	0.183425073	0.1867	0.9
0.362	0.8154	0.791416882	0.8052	1.4
0.6401	1.4418	1.399408691	1.422	2.1
0.9182	2.0683	2.0074005	2.036	2.9
1.1963	2.6947	2.61539231	2.645	3.3
1.4744	3.3212	3.223384119	3.248	3.9
1.7526	3.9476	3.831594551	3.845	4.5
2.0307	4.574	4.439586361	4.433	5.1
2.3088	5.2005	5.04757817	5.013	5.7
2.5869	5.8269	5.655569979	5.582	6.4
2.865	6.4534	6.263561788	6.142	7.2
3.1431	7.0798	6.871553597	6.696	7.7
3.4212	7.7063	7.479545406	7.25	8.2
3.6994	8.3327	8.087755839	7.756	8.6
3.9775	8.9592	8.695747648	8.3931	9.2
4.2556	9.5856	9.303739457	8.764	9.8
4.5337	10.212	9.911731267	9.245	10.2
4.8118	10.8385	10.51972308	9.72	10.6

Fig. 6 Raw Data

MATLAB Code for FEA: